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CONNECTION PRESERVING ACTIONS OF LATTICES IN $SL_n\mathbb{R}$

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ABSTRACT

We consider connection-preserving actions of lattice subgroups of $SL_n\mathbb{R}$ on compact Riemannian manifolds, for $n \geq 3$. The main result gives a description of such actions, when $\dim M = n+1$, as follows: Either the action preserves a smooth invariant Riemannian metric on M or it can be described in terms of linear actions on the n-torus.

0. Introduction

Let G be a simple Lie group of $\mathbb{R}$-rank≥ 2 and Γ a lattice subgroup of G. Assume that Γ acts on a smooth, compact manifold M via diffeomorphisms that preserve some volume density. Also assume that the action is not finite, i.e. it does not factor through a finite quotient of Γ. For such setting, R. Zimmer ([Z3]) asked the following question: Can the action of Γ be described in algebraic terms, or do there exist genuinely (differential) topological examples?

This issue of **rigidity** of actions of higher rank lattices has seen much recent progress, and we mention in particular the papers [H], and [KL1], [KL2], which bring into play ideas from the theory of hyperbolic dynamical systems.

In the present paper we consider actions of lattice subgroups of $SL_n\mathbb{R}$ on Riemannian manifolds and the main assumption we explore is that the group acts by connection-preserving diffeomorphisms. The principal ingredient in our argument is Zimmer's superrigidity theorem for cocycles and one important step in the proof is to obtain a smooth version of that theorem under our assumptions. Before stating our results we point to the following theorem proven in [Z1].

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THEOREM 0.1: *Let Γ be a lattice in $\mathrm{SL}_n\mathbb{R}$, $n \geq 3$.*

(1) *Let M be a compact manifold with $\dim M < n$. Then any action of Γ on M that preserves a volume density and a connection is finite, i.e. is an action by a finite quotient of Γ.*
(2) *Let M be a compact, connected, Riemannian manifold with $\dim M = n$. Suppose Γ acts on M by volume preserving, affine transformations. If the action is not finite, M is flat and Γ is commensurable to (a conjugate of) $\mathrm{SL}_n\mathbb{Z}$. In particular, if Γ is co-compact, any volume preserving affine action on a compact Riemannian n-manifold is finite.*

The main result in the present paper extends the above theorem as it also accounts for $n+1$-dimensional manifolds. We show:

THEOREM 0.2: *Let M be a compact, connected, Riemannian manifold and Γ a lattice in $\mathrm{SL}_n\mathbb{R}$, $n \geq 3$. Let us assume that Γ acts on M as a group of C^∞ diffeomorphisms preserving the Levi-Civita connection. We assume moreover that the action does not preserve any smooth Riemannian metric. Then, by possibly having to pass to a finite index subgroup of Γ, the following holds:*

(1) *If $\dim M = n$, there exists a finite Riemannian covering M' of M, a finite extension Γ' of Γ, and a smooth affine action of Γ' on M' lifting the Γ-action equivariantly, so that the latter (the Γ' action on M') is smoothly conjugate to a linear action of Γ' on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The diffeomorphism $\varphi : \mathbb{T}^n \to M'$ that conjugates the two actions is affine and M is itself a flat torus.*
(2) *If $\dim M = n+1$, there exists as before an action of Γ' on M' that lifts the action of Γ on M to a finite covering M' of M, which is smoothly conjugate to a linear action on $\mathbb{T}^{n+1}$. M is a flat manifold fibered over a flat torus, whose fibers form a Γ-invariant family of closed geodesics. Therefore the action factors through the quotient $M/\mathbb{T}^1$, to which the description in (1) applies.*

In fact, more can be said: If Γ acts isometrically and the action is not finite, then $\dim(\mathrm{Iso}(M)) \geq n^2 - 1$ (see [Z3]). Our main concern, here, is the case when the action of Γ exhibits some form of hyperbolic behaviour, in which case no metric can be preserved.

Theorem 0.2 will follow after we establish the proposition given below. We denote by $T\gamma$ the derivative map of a function γ and $X^\gamma = (T\gamma)X \circ \gamma^{-1}$, for a vector field X, and a diffeomorphism γ of M. We shall say that an affine connection ∇ has **bounded parallel transport** if the operator describing parallel transport along any path is bounded in norm by a (path independent) constant.

PROPOSITION 0.3: *Let M be a compact, connected manifold and Γ a lattice in $\mathrm{SL}_n\mathbb{R}$, $n \geq 3$. Assume that Γ acts on M as a group of C^∞ diffeomorphisms preserving a C^1 torsion free connection ∇ with bounded parallel transport. Assume moreover that the action does not preserve a Riemannian metric. Then the following holds:*

(1) *If $\dim M = n$, M is a ∇-flat torus and the action is linear in the following sense: There exists (i) a finite covering M' of M (of order $\leq 2^n$) and a finite extension Γ' of Γ that lifts equivariantly the action of Γ to M', (ii) a frame of commuting C^1 vector fields $X_1, \dots, X_n$ on M', (iii) an automorphism a of $\mathrm{SL}_n\mathbb{R}$, (iv) a subgroup Γ'' of Γ' of index ≤ 2 such that for all $\gamma \in \Gamma''$,*

$$X_i^\gamma = \sum_{j=1}^{n} a_{ji}(\gamma) X_j.$$

Moreover the action of Γ on M has a fixed point.

(2) *If $\dim M = n+1$, we have: (i) M is ∇-flat and it is a fiber bundle over a flat n-torus whose fibers are geodesically imbedded circles $\mathbb{T}^1$. The action maps fibers to fibers, so it factors through the quotient $M/\mathbb{T}^1$, a flat n-torus. M possesses a Γ-invariant, n-dimensional foliation transversal to the foliation by circles, consisting of flat leaves and carrying a C^1 smooth, Γ-invariant, transverse measure. If M is orientable, it is a flat $n+1$-dimensional torus. (ii) The action is linear in the following sense: There exists a finite covering M' of M of order $\leq 2^{n+1}$, commuting, C^1 vector fields $X_1, \dots, X_{n+1}$ on M' which are parallel with respect to the connection lifted from M, a finite extension Γ' of Γ (by the group of deck transformations) that acts on M' equivariantly and a group homomorphism $A : \Gamma' \to \mathrm{SL}_n\mathbb{R}$ of the form*

$$A = (A_{ij}) = \begin{pmatrix} \epsilon_1 a & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

where a is an automorphism of $\mathrm{SL}_n\mathbb{R}$ and ϵ_1 and ϵ_2 are homomorphisms into $\{+1, -1\}$, so that for any $\gamma \in \Gamma'$

$$X_i^\gamma = \sum_{j=1}^{n+1} A_{ji}(\gamma) X_j.$$

Moreover, by possibly having to pass to a finite index subgroup of Γ, the action of Γ on M has a fixed point.

Remarks 0.4: 1. We expect that a result similar to Theorem 0.2 and Proposition 0.3 should hold for $\dim M < N$, where N is the minimum integer $> n$ equal to the dimension of some representation of $\mathrm{SL}_n\mathbf{R}$ (the fibers of the foliation over M would be totally geodesic, compact submanifolds such that the quotient would still be a flat torus). In fact, most of the argument used below applies to this case and only at the end, in section 4, we use $k = 1$ in order to characterize the holonomy maps of that foliation as the Poincaré's return map of a flow. This device introduces technical simplifications but does not seem to be essential to the proof.

2. The reason for assuming that the action preserves a Riemannian connection (or a connection with bounded parallel transport, as in the proposition) instead of a more general affine connection can be seen in Lemma 2.4. We believe that the property of having bounded parallel transport is automatically verified under the assumptions of the theorem. We plan to expand on this remark in a future work.

3. Many of the ideas employed here can be made to work for a C^0 connection, and it is natural to ask whether the theorem could be proven under this weaker assumption. Thus, flatness would have to be characterized in terms of the holonomy group and the linearizing frame in Proposition 0.3 would be only continuous. From such a C^0 frame one could try to derive (by an argument perhaps similar to the one shown at the end of this section) a C^0 conjugacy between the initial action and a linear action. Results in [H] or [KL2] then will imply that the conjugacy is actually C^∞.

4. Another question that arises is whether the n-dimensional foliation, referred to in Proposition 0.3 item (2), has a closed leaf. In that case it would follow from the proof of the proposition that all leaves are closed. One could then try to show that M is covered by a product $\mathbb{T}^n \times \mathbb{T}^1$ and that the action decomposes as a product (Γ acting finitely on $\mathbb{T}^1$).

5. At moments along the proof there arises the need to orient certain line fields and find fixed points for the actions. These are the main reason for complicating the statements of the above theorems by introducing finite coverings, finite extensions of Γ, and subgroups of finite index. Perhaps a more careful analysis will render some of these details unnecessary. ∎

We conclude this section with a simple remark that shows how Theorem(0.2) is reduced to Proposition(0.3).

Denote by φ_t^Y: $M' \to M'$ the flow of a vector field Y on M'. Since the C^1 vector fields X_i are linearly independent and commute, we obtain a locally free action of $\mathbb{R}^{n+1}$ on M':

$$\varphi : \mathbb{R}^{n+1} \times M' \to M'$$

$$\varphi(t_1, \dots, t_{n+1}, x) = \varphi_{t_1}^{X_1} \circ \cdots \circ \varphi_{t_{n+1}}^{X_{n+1}}(x)$$

By choosing $x_0 \in M$ and considering its $\mathbb{R}^{n+1}$-orbit, we obtain a covering map

$$p : \mathbb{R}^{n+1} \to M', \quad p(t_1, \dots, t_{n+1}) = \varphi(t_1, \dots, t_{n+1}, x_0)$$

Since M' is compact, the isotropy subgroup Λ of x_0 is a lattice of $\mathbb{R}^{n+1}$, hence it is isomorphic to $\mathbb{Z}^{n+1}$. We thus obtain a diffeomorphism $\mathbb{T}^{n+1} \simeq \mathbb{R}^{n+1}/\Lambda \to M$. If we choose x_0 to be a fixed point for the action, then the diffeomorphism constructed above is the conjugating diffeomorphism claimed in Theorem 0.2.

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1. Oseledec's multiplicative ergodic theorem

One main ingredient in our proof is Oseledec's theorem. The following version is taken from [W, Theorem 10.4, p.234].

Given a differentiable manifold M and a homeomorphism f: $M \to M$, denote by $\mathcal{B}$ the σ-algebra of Borel subsets of M and by $\mathcal{M}(M, f)$ the set of all f-invariant probability measures on $\mathcal{B}$.

PROPOSITION 1.1: *Let f: $M \to M$ be a C^1 diffeomorphism of a compact, smooth manifold M of dimension m. Choose a Riemannian metric on M and let $\|\cdot\|$ denote its norm. There exists $\Lambda \in \mathcal{B}$ such that $f(\Lambda) = \Lambda$, $\mu(\Lambda) = 1$ for all $\mu \in \mathcal{M}(M, f)$, and the following properties hold: There exist measurable, f-invariant functions s: $\Lambda \to \{1, \dots, m\}$, χ_i: $\Lambda \to \mathbb{R}$ where $1 \le i \le s$, and a measurable decomposition*

$$TM|_\Lambda = E_1 \oplus \cdots \oplus E_s$$

of the tangent bundle over Λ into f-invariant subbundles ($Tf \cdot E_i = E_i \circ f$) such that for all $x \in \Lambda$, all $v \in E_i(x) \setminus \{0\}$, $1 \le i \le s(x)$, we have

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Tf_x^n v\| = \chi_i(x).$$

The above decomposition and the functions χ_i are unique and do not depend on the Riemannian metric chosen.

The above decomposition is called the **Oseledec decomposition** of f and the functions χ_i, its **Lyapunov exponents**.

It can be shown that if instead of f we have k commuting C^1diffeomorphisms $f_1, \ldots, f_k$, there exists a Oseledec decomposition common to all of them, in the following sense. Denote by $\mathcal{A}$ the group generated by $\{f_1, \ldots, f_k\}$ and by $\mathcal{M} = \mathcal{M}(M, \mathcal{A})$ the set of probability measures on $\mathcal{B}$ invariant under each f_i. Then there exists $\Lambda \in \mathcal{B}$, $f(\Lambda) = \Lambda$ for all $f \in \mathcal{A}$, $\mu(\Lambda) = 1$ for all $\mu \in \mathcal{M}$ and a measurable decomposition

$$TM|_\Lambda = E_1 \oplus \cdots \oplus E_s$$

such that for each $f \in \mathcal{A}$ having Oseledec decomposition $TM = F_1 \oplus \cdots \oplus F_k$ we have

$$F_j = \bigoplus_{k \in I_j} E_k, \quad I_j \subset \{1, \ldots, s\}.$$

It will be helpful also to view Oseledec's theorem in a form that applies to cocycles. First we give a few definitions. We denote by $F(M)$ the bundle of all linear frames on TM. Each frame will be regarded as a linear isomorphism $\alpha\colon \mathbb{R}^m \to TM_x$ for some x in M. The base point projection $p\colon F(M) \to M$, $p(\alpha) = x$, defines a $\mathrm{GL}_n\mathbb{R}$-principal bundle. Let $\mu \in \mathcal{M}(M, f)$ and $\sigma\colon M \to F(M)$ a Borel section of this bundle. Define a cocycle $A\colon \mathbb{Z} \times M \to \mathrm{GL}_m\mathbb{R}$ as follows:

$$Tf_x^k \circ [\sigma(x)] = \sigma(f^k x) \circ A(k, x).$$

Since f is a diffeomorphism of a compact manifold, we can choose σ so that the positive parts $\log^+ \|A(\pm 1, \cdot)\|$ of $\log \|A(\pm 1, \cdot)\|$ are integrable, i.e. belong to $L^1(\mu, \mathcal{B}, M)$. It now follows from Oseledec's theorem, as stated in [R], that there exists an f-invariant set $\Lambda \in \mathcal{B}, \mu(\Lambda) = 1$, and for each $x \in \Lambda$ a decomposition

$$\mathbb{R}^m = W_1(x) \oplus \cdots \oplus W_{s(x)}(x)$$

that depends measurably on x, for which

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|A(n, x)u\| = \chi_i(x)$$

for $0 \neq u \in W_i(x)$. Moreover $A(m,x)W_i(x) = W_i(f^m x)$ for all $x \in \Lambda$. The spaces $E_i(x) = \sigma(x)W_i(x) \subset TM_x$ are the ones given in the previous proposition. Note that if A takes values in a compact subgroup of $GL_m\mathbb{R}$, all Lyapunov exponents vanish.

Suppose that the cocycle A has the form

$$A = \begin{pmatrix} A^{(1)} & * \\ 0 & A^{(2)} \end{pmatrix}$$

where $A^{(1)}$ and $A^{(2)}$ are cocycles taking values in $GL_k\mathbb{R}$ and $GL_{m-k}\mathbb{R}$ respectively. Then it can be shown that the Lyapunov exponents of A are those of $A^{(1)}$ and $A^{(2)}$.

At one point we shall need to consider a section $\sigma: M \to F(M)$ whose associated cocycle A may have possibly nonintegrable $\log^+ \|A(\pm 1, \cdot)\|$. The following lemma will be needed.

LEMMA 1.2: *Let M be a compact, smooth manifold, $f: M \to M$ a C^1 diffeomorphism and $\mu \in \mathcal{M}(M,f)$. Let $\sigma: M \to F(M)$ be a Borel section of the frame bundle over M and $A: \mathbb{Z} \times M \to GL_m\mathbb{R}$ the associated cocycle for the $\mathbb{Z}$-action defined by f. Assume that there exists an f-invariant set $\Omega \in \mathcal{B}, \mu(\Omega) = 1$ and, for each $x \in \Omega$, a decomposition*

$$\mathbb{R}^m = W_1(x) \oplus \cdots \oplus W_{s(x)}(x)$$

such that

$$\chi_i(x) \stackrel{\text{def}}{=} \lim_{n \to \pm\infty} \frac{1}{n} \log \|A(n,x)u\|$$

exists and is the same for each $u \in W_i(x) \setminus \{0\}$. Then, defining $E_i(x) = \sigma(x)W_i(x)$, we have: The Oseledec decomposition for f agrees μ-almost everywhere with $E_1 \oplus \cdots \oplus E_s$ and the Lyapunov exponent of vectors in E_i is $\chi_i, i = 1, \ldots, s$.

Proof: The issue here is simply to verify that for μ-a.e. x and all $u \in W_i(x) \setminus \{0\}$

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Tf_x^n \sigma(x)u\| = \chi_i(x),$$

knowing that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|A(n,x)u\| = \chi_i(x).$$

For each $\epsilon > 0$ we can find a subset $\Omega_\epsilon \subset \Omega$ such that $\mu(\Omega_\epsilon) > 1 - \epsilon$ and $\sigma|_{\Omega_\epsilon}$ takes values in a compact subset of $F(M)$. In particular, on Ω_ϵ, σ is bounded in the following sense: With respect to some Riemannian metric on M with norm $\|\cdot\|$ (and denoting by $\|\cdot\|_e$ the Euclidian norm in $\mathbb{R}^m$), there exists a constant $c \geq 1$ such that for all $u \in \mathbb{R}^m$ and $x \in \Omega_\epsilon$

$$\frac{1}{c}\|u\|_e \leq \|\sigma(x)u\| \leq c\|u\|_e.$$

Set $\overline{\Omega}_\epsilon = \Omega_\epsilon \cap \Lambda$, where Λ is given in Proposition (1.1). For any $x \in \overline{\Omega}_\epsilon$ and $u \in W_i(x) \setminus \{0\}$ the limits

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Tf_x^n \sigma(x)u\|$$

exist as is assured by Oseledec's theorem (note that the limits exist for all $u \in \mathbb{R}^m \setminus \{0\}$). So it suffices to verify their values for some subsequence $n_i \to \pm\infty$ as $i \to \pm\infty$. According to Poincaré's recurrence Theorem, for a.e. $x \in \overline{\Omega}_\epsilon$ there exists a doubly infinite sequence $\{n_i : i \in \mathbb{Z}\}$ such that $f^{n_i}(x) \in \overline{\Omega}_\epsilon$. Now, by definition of A,

$$\frac{1}{c}\|A(n,x)u\| \leq \|Tf_x^{n_i}\sigma(x)u\| \leq c\|A(n,x)u\|.$$

This implies

$$\left|\frac{1}{n_i}\log\|Tf_x^{n_i}\sigma(x)u\| - \frac{1}{n_i}\log\|A(n_i,x)u\|\right| \leq \frac{1}{n_i}\log c,$$

from which (and since $\epsilon > 0$ is arbitrary) the lemma follows. ∎

2. Invariant connections

Let M be a smooth manifold and $p\colon E \to M$ a smooth real vector bundle over M of rank k. By an **automorphism** of E we mean a diffeomorphism $\overline{f}$ of E that maps fibers into fibers and the restriction to each fiber is a linear isomorphism. In particular there is a diffeomorphism f of M such that $p \circ \overline{f} = f \circ p$.

Denote by $C^r(E)$ the space of C^r sections of E and consider a connection

$$\nabla\colon C^{r+1}(E) \to C^r(T^*M \otimes E).$$

We shall use the following notation: Given $X \in C^r(TM)$, $\xi \in C^{r+1}(E)$, and $\overline{f}$ an automorphism of E covering the diffeomorphism $f\colon M \to M$, we write

$$X^f = (Tf)\circ X \circ f^{-1},\quad \xi^f = \overline{f}\xi \circ f^{-1},\quad (\nabla^f)_X\xi = (\nabla_{X^f}\xi^f)^{f^{-1}}.$$

An automorphism $\overline{f}$ is said to be **affine** if $\nabla^f = \nabla$.

Denote by $F(E)$ the $GL_k\mathbb{R}$-principal bundle of frames of E, a frame $\alpha \in F(E)$ being a linear isomorphism $\alpha: \mathbb{R}^k \to E_x$, $x \in M$. The base point map will be denoted $p: F(E) \to M$, $p(\alpha) = x$. Let Γ be a group of automorphisms of E and let $\sigma: M \to F(E)$ be a section of the frame bundle. Let $A: \Gamma \times M \to GL_k\mathbb{R}$ denote the corresponding cocycle, so that for all $\gamma \in \Gamma$ and $x \in M$

$$\sigma^\gamma(x) = \overline{\gamma} \circ \sigma(\gamma^{-1}x) = \sigma(x) \circ A(\gamma, \gamma^{-1}x),$$

(the first equality defines σ^γ and the second one defines A) and the cocycle relation

$$A(\gamma_1\gamma_2, x) = A(\gamma_1, \gamma_2 x) \circ A(\gamma_2, x)$$

holds.

LEMMA 2.1: *Let M be a connected, smooth manifold, $E \to M$ a smooth vector bundle over M of rank k, and ∇ a connection on E. Let Γ be a group of affine automorphisms of E and σ: $M \to E$ a C^1, ∇-parallel section of $F(E)$. Then there exists a group homomorphism A: $\Gamma \to GL_k\mathbb{R}$ such that for any $\gamma \in \Gamma$, $\sigma^\gamma = \sigma \circ A(\gamma)$.*

Proof: We need only check that the cocycle A does not depend on x. For any $X \in C^\infty(TM)$, write $Y = X^{\gamma^{-1}}$, $\gamma \in \Gamma$. Then, if we set $A_\gamma(x) = A(\gamma, \gamma^{-1}x)$ and $\mathcal{L}_X A_\gamma =$ the Lie derivative of A_γ along X,

$$\begin{aligned} 0 = (\nabla_Y \sigma)^\gamma = \nabla_X \sigma^\gamma &= \nabla_X(\sigma \circ A_\gamma), \\ &= (\nabla_X \sigma) \circ A_\gamma + \sigma \circ \mathcal{L}_X A_\gamma, \\ &= \sigma \circ \mathcal{L}_X A_\gamma. \end{aligned}$$

Therefore $\mathcal{L}_X A_\gamma = 0$ for all X. So, as M is connected, A_γ is constant. ∎

Let ∇ be a C^r $(r \geq 0)$ connection on TM, for a smooth manifold M. Given a C^1 path $l: [t_1, t_2] \to M$, denote by $P_l: TM_{l(t_1)} \to TM_{l(t_2)}$ the parallel trasport operator along l. We say that ∇ has **bounded parallel transport** if, for some Riemannian metric on M, there exists a constant $C \geq 1$ such that for any C^1 path l and vector $Z \in TM_{l(t_1)}$, we have $\|P_\alpha Z\| \leq C\|Z\|$. It immediately follows that

$$\frac{1}{C}\|Z\| \leq \|P_\alpha Z\| \leq C\|Z\|.$$

(Of course, if ∇ is Riemannian we can take $C = 1$ as P is then an isometry.)

We recall that a density on an m-dimensional manifold M is locally the absolute value of an m-form.

LEMMA 2.2: *Let M be a compact, connected, smooth manifold and Γ a group of diffeomorphisms of M. Assume that the diffeomorphisms are affine for a C^r $(r \geq 0)$ connection ∇ on TM having bounded parallel transport. Then Γ preserves a ∇-parallel volume density which is C^r-differentiable.*

Proof: Consider the line bundle $p\colon E = \Lambda^m(T^*M) \to M$, $m = \dim M$. Given $\nu \in E \setminus \{0\}$, $x = p(\nu)$, and any C^1 loop $l\colon [0,1] \to M$ based at x, we claim that

$$P_l \nu = \pm \nu.$$

In fact, define c by the equation $P_l\nu = c\nu$. Then if l_n denotes the loop that turns n times around l, we have $P_{l_n}\nu = c^n\nu$. But since ∇ has bounded parallel transport, we must have $c = \pm 1$. In this way, we can define a density ρ on M by parallel transporting ν to all other points in M. By construction ρ is ∇-parallel. Lemma 2.1 implies that for all $f \in \Gamma$, $\rho^f = c'\rho$, where $c' = c'(f)$ is a positive constant. But since M is compact and f a diffeomorphism,

$$0 < \int_M \rho^f = \int_M \rho < \infty,$$

hence $c' = 1$. That ρ is C^r follows from the fact that parallel transport is locally determined by a linear system of O.D.E.s with C^r coefficients (the Cristoffel symbols of the C^r connection) and one has C^r dependence of solutions on parameters. ∎

LEMMA 2.3: *Let $f\colon M \to M$ be a diffeomorphism preserving a C^0 connection ∇ on TM. Let $l\colon [t_1, t_2] \to M$ be a C^1 path. Then*

$$Tf_{l(t_2)} \circ P_l = P_{f\circ l} \circ Tf_{l(t_1)}.$$

Proof: Follows immediately from $\nabla^f = \nabla$. ∎

LEMMA 2.4: *Let $f\colon M \to M$ be a diffeomorphism of a connected, compact, smooth manifold M. Assume that f preserves a C^r connection ∇ $(r \geq 0)$ with bounded parallel transport. Then the Oseledec decomposition of f is defined everywhere and is ∇-parallel. In particular, it is a C^r-decomposition.*

Proof: Let $x \in \Lambda$, Λ as in Proposition (1.1). Let $\chi_1, \ldots, \chi_s$ be the exponents at x and $TM_x = E_1(x) \oplus \cdots \oplus E_s(x)$ the Oseledec decomposition at x. Let y be any other point in M and l: $[t_1, t_2] \to M$ a smooth path connecting x to y. Denote by $E_i(y)$ the parallel transport of $E_i(x)$ along l. Then for every $Z \in E_i(y)$, $Z \neq 0$, we can find $Z' \in E_i(x) \setminus \{0\}$ such that, for all integers n

$$Tf_y^n Z = Tf_y^n P_l Z' = P_{f^n \circ l} Tf_x^n Z'$$

(where the second equality is due to Lemma 2.3). Since ∇ has bounded parallel transport, there exists $C \geq 1$ such that for all integers n

$$\frac{1}{C}\|Tf_x^n Z'\| \leq \|P_{f^n \circ l} Tf_x^n Z'\| \leq C\|Tf_x^n Z'\|.$$

Therefore $\frac{1}{C}\|Tf_x^n Z'\| \leq \|Tf_y^n Z\| \leq C\|Tf_x^n Z'\|$, from which we obtain

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Tf_y^n Z\| = \chi_i.$$

In particular, the parallel transport of $E_i(x)$ to y does not depend on the path chosen between x and y since the characterization of $E_i(y)$ given by the last equation does not involve the path. To establish C^r differentiability we invoke, as in Lemma 2.2, the C^r dependence of solutions of O.D.E.s on parameters. ∎

3. Zimmer's superrigidity theorem for cocycles

As before, we denote by p: $F(M) \to M$ the frame bundle over a smooth, m-dimensional manifold M. If Γ is a group of diffeomorphisms of M, it also acts by automorphisms of $F(M)$:

$$(\gamma, \sigma) \in \Gamma \times F(M) \to T\gamma_x \circ \sigma \in F(M),$$

where $x = p(\sigma)$ and we recall that σ is viewed as a linear isomorphism $\mathbb{R}^m \to TM_x$. Let A: $\Gamma \times M \to \mathrm{GL}_m\mathbb{R}$ denote the cocycle of an action of Γ on M, obtained by considering a Borel section σ: $M \to F(M)$. If the action of Γ preserves a (measurable) subbundle E of TM of dimension l and σ is a section of p adapted to E (i.e., for almost every $x \in M$, $(\sigma(x)e_1, \ldots, \sigma(x)e_l)$ spans E, where $(e_1, \ldots, e_m)$ is the standard basis of $\mathbb{R}^m$), then the associated cocycle has the form

$$A(\gamma, x) = \begin{pmatrix} A^{(1)}(\gamma, x) & * \\ 0 & A^{(2)}(\gamma, x) \end{pmatrix}$$

where $A^{(1)}$ and $A^{(2)}$ are cocycles taking values in $\mathrm{GL}_l\mathbb{R}$ and $\mathrm{GL}_{m-l}\mathbb{R}$, respectively. Conversely if for some section of p the cocycle takes the above form, there exists a Γ-invariant subbundle of dimension l, given by

$$x \in M \to \sigma(x)L, \quad L = \mathrm{span}\{e_1, \ldots, e_l\}.$$

If M is orientable and ν is a volume form preserved by Γ, then Γ acts as a group of automorphisms of the $\mathrm{SL}_m\mathbb{R}$-principal bundle $P \subset F(M)$ consisting of all $\sigma \in F(M)$ for which $\nu(\sigma e_1, \ldots, \sigma e_m) = 1$. Let μ be an ergodic invariant measure for the action of Γ. Two sections σ and σ' of P are said to be equivalent if there is a Borel function φ: $M \to \mathrm{SL}_m\mathbb{R}$ for which $\sigma' = \sigma \circ \varphi$ at μ-almost every point. Two cocycles are said to be equivalent if they are associated to equivalent sections. It can be shown (see [Z2], where a more general statement is proven) that an algebraic $\mathbb{R}$-group $L \subset \mathrm{SL}_m$ exists with the following property: A is equivalent to a cocycle taking values in $L_{\mathbb{R}}$ but there is no equivalent cocycle taking values in $L'_{\mathbb{R}}$ for a $\mathbb{R}$-group L' which is a proper $\mathbb{R}$-subgroup of L. $L_{\mathbb{R}}$ is unique up to conjugacy in $\mathrm{SL}_m\mathbb{R}$ and its conjugacy class is the **algebraic hull** of the cocycle. The algebraic hull is **compact** if any of its representatives $L_{\mathbb{R}}$ is. The algebraic hull for the invariant volume ν can be defined as a map from the ergodic components of ν to conjugacy classes of algebraic subgroups of $\mathrm{SL}_m\mathbb{R}$.

The following proposition is a consequence of the superrigidity theorem for cocycles ([Z2], [Z3]).

PROPOSITION 3.1:

(1) *Let M be an oriented, compact, smooth manifold of dimension $m = n + k$, $k = 0, 1$. Let ν be a volume form and Γ a lattice in $\mathrm{SL}_n\mathbb{R}, n \geq 3$. Assume that Γ acts on M via diffeomorphisms that preserve ν. Let μ be an ergodic component of the measure defined by ν and A: $\Gamma \times M \to \mathrm{SL}_m\mathbb{R}$ the cocycle for Γ associated to some measurable section of P. Assume that the algebraic hull of A is not compact. Then there exists a measurable section of P, σ: $M \to P$, and an automorphism a of $\mathrm{SL}_n\mathbb{R}$ such that for μ-almost every $x \in M$ and all $\gamma \in \Gamma$, the cocycle associated with σ has either one of the following two forms:*

$$A_\sigma(\gamma, x) = \begin{pmatrix} \epsilon(\gamma, x)a(\gamma) & B(\gamma, x) \\ 0 & \overline{A}(\gamma, x) \end{pmatrix} \text{ or } \begin{pmatrix} \overline{A}(\gamma, x) & B(\gamma, x) \\ 0 & \epsilon(\gamma, x)a(\gamma) \end{pmatrix},$$

where $\overline{A}$ is a cocycle taking values in a compact subgroup of $\mathrm{GL}_k\mathbb{R}$, ϵ a cocycle taking values in $\{\pm 1\}$, and a an automorphism of $\mathrm{SL}_n\mathbb{R}$.

(2) *Consider the same conditions as in part (1), except that the algebraic hull of A is now assumed compact. Assume moreover that the diffeomorphisms in* Γ *are affine for a* C^0 *connection* ∇. *Then the action preserves a smooth invariant metric.*

In order to state next lemma we need a few remarks.

(1) A theorem in [P-R] implies that, for any lattice $\Gamma \subset SL_n\mathbb{R}$, one can find a Cartan subgroup H of $SL_n\mathbb{R}$ for which $H/H \cap \Gamma$ is compact. It follows that there exists a free abelian group $\mathcal{A} \subset \Gamma$ of rank $n-1$ whose elements are simultaneously diagonalizable over $\mathbb{R}$.

(2) The group of automorphisms of $SL_n\mathbb{R}$ is generated by the conjugations $g \to g^h = hgh^{-1}$, $h \in SL_n\mathbb{R}$ and $a(g) = (g^t)^{-1}$. In particular, if $\mathcal{A}$ is a free abelian subgroup of $SL_n\mathbb{R}$ of rank $n-1$ which is diagonalizable over $\mathbb{R}$, so is $a(\mathcal{A})$ for any automorphism a.

LEMMA 3.2: *Let* $\{v_1, \dots, v_n\}$ *be a basis of* $\mathbb{R}^n$ *consisting of simultaneous eigenvectors of* $a(\mathcal{A})$, *where* a *is some automorphism of* $SL_n\mathbb{R}$. *Define* $\lambda_i\colon \mathcal{A} \to \mathbb{R}\setminus\{0\}$ *by the equation* $a(\gamma)v_i = \lambda_i(\gamma)v_i$ *for all* $\gamma \in \mathcal{A}$. *Then*

(1) $\lambda_1 \cdot \dots \cdot \lambda_n = 1$

(2) *Given* $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, *define*

$$\lambda^m(\gamma) = \lambda_1^{m_1}(\gamma) \cdot \dots \cdot \lambda_n^{m_n}(\gamma).$$

Then given finitely many integer vectors $m^{(1)}, \dots, m^{(r)}$, *all different from* $\pm(1, \dots, 1)$, *and any permutation* $s\colon \{1, \dots, n\} \to \{1, \dots, n\}$, *we can find* $\gamma \in \mathcal{A}$ *such that* $\lambda^{m^{(i)}}(\gamma) \neq \pm 1$, *for all* $1 \le i \le n$ *and*

$$1 < \lambda_{s(1)}(\gamma) < \dots < \lambda_{s(n-1)}(\gamma), \quad 1 > \lambda_{s(n)}(\gamma).$$

Proof: (1) follows from det $a(\gamma) = 1$ and (2) from the co-compactness of $\mathcal{A}$ in some Cartan subgroup of $SL_n\mathbb{R}$. ∎

Let $\sigma\colon M \to P$ be the section of the bundle P obtained in Proposition 3.1 (we assume that the matrix on the left occurs; the other case is similar) and v_i as in the previous lemma. Define the measurable vector fields $X_i(x) = \sigma(x)v_i$, $1 \le i \le n$. Then for all $\gamma \in \mathcal{A}$ we have

$$T\gamma_x X_i(x) = \epsilon(\gamma, x)\lambda_i(\gamma)X_i(\gamma x)$$

where $\epsilon \in \{\pm 1\}$. In fact,

$$\begin{aligned} T\gamma_x X_i(x) &= T\gamma_x \sigma(x) v_i = \sigma(\gamma x) \circ A(\gamma, x) v_i \\ &= \sigma(\gamma x)\epsilon(\gamma, x) a(\gamma) v_i = \epsilon(\gamma, x)\lambda_i(\gamma)\sigma(\gamma x) v_i \\ &= \epsilon(\gamma, x)\lambda_i(\gamma) X_i(\gamma x). \end{aligned}$$

In particular, applying Proposition 1.1, we obtain the following lemma.

LEMMA 3.3: *Assume the conditions of Proposition 3.1, item (1). The action of $\mathcal{A}$ on M has Oseledec decomposition given by $E_1 \oplus \cdots \oplus E_{n+1}$, where $E_i = \mathbb{R}X_i$ for $i = 1, \cdots, n$ and E_{n+1} is a k-dimensional subbundle. Moreover for all $\gamma \in \mathcal{A}$ and μ-almost every x, we have*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|T\gamma_x^n X_i(x)\| = \log |\lambda_i(\gamma)|,$$

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|T\gamma_x^n Z\| = 0, \quad \forall Z \in E_{n+1}(x) \setminus \{0\}.$$

LEMMA 3.4:

(1) *Let M be a connected, compact, smooth manifold of dimension $m = n + k$, $k = 0, 1$. Let Γ be a lattice in $\mathrm{SL}_n\mathbb{R}, n \geq 3$. Assume that Γ acts on M by affine diffeomorphisms for a C^r, torsion-free connection ∇ ($r \geq 0$) with bounded parallel transport. If the action does not preserve a smooth Riemannian metric, there exists a C^r decomposition of TM, $TM = E_1 \oplus \cdots \oplus E_{n+1}$, which agrees with the one in Lemma (3.3) almost everywhere. Here we have $\dim E_i = 1$ for $i = 1, \cdots, n$, $\dim E_{n+1} = k$, all subbundles are ∇-parallel and $\mathcal{A}$-invariant. The subbundle E defined as $E = E_1 \oplus \cdots \oplus E_n$ is Γ-invariant.*

(2) *If we also assume that $k = 1$ and $r \geq 1$, the following holds: There exists a finite covering M' of M, whose group $\mathcal{D}$ of deck transformations has cardinality $|\mathcal{D}| \leq 2^{n+1}$, and a frame of C^r vector fields $X_1, \cdots, X_{n+1}$ defined on M' which are ∇-parallel (for the connection lifted from M, still denoted ∇) so, in particular, M is ∇-flat. If Γ' denotes the finite extension of Γ by $\mathcal{D}$, acting equivariantly on M' by ∇-affine diffeomorphisms and $\sigma\colon M' \to F(M')$ the section of $F(M')$ defined by $\sigma(x)e_i = X_i$, then there exists a homomorphism $A : \Gamma' \to \mathrm{GL}_{n+1}\mathbb{R}$ so that*

$$T\gamma_x \circ [\sigma(x)] = \sigma(x) \circ [A(\gamma)],$$

for all $x \in M'$ and $\gamma \in \Gamma'$. Moreover, after a change of basis, A has the form

$$A(\gamma) = \begin{pmatrix} \epsilon_1(\gamma)a(\gamma) & 0 \\ 0 & \epsilon_2(\gamma) \end{pmatrix},$$

where a is an automorphism of $SL_n\mathbb{R}$ and ϵ_1 and ϵ_2 are homomorphisms into $\{\pm 1\}$.

(3) *The vector fields $X_1, \ldots, X_{n+1}$ in item (2) commute: $[X_i, X_j] = 0$ for all i and j.*

Proof: Item (1) follows from Lemmas 2.4 and 3.3. As for item (2), let $X_i(x)$ be any nonzero vector in $E_i(x)$. Since E_i is parallel, for any C^1 loop l: $[0,1] \to M$ based at x, $P_l X_i(x) = c_i(l) X_i(x)$ for some number $c_i(l)$. Since ∇ has bounded parallel transport, $c_i(l) = \pm 1$ and, by continuity, the value depends only on the homotopy class $[l]$. Therefore we obtain a group homomorphism

$$(c_1, \ldots, c_{n+1})\colon \pi_1(M, x) \to \underbrace{\mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}}_{n+1 \text{ times}}$$

The kernel of this homormorphism defines a finite normal covering of M where $X_i(x)$ can be extended to a globally defined vector field using parallel transport (note that X_i is defined on M only up to sign). Lemma 2.1 implies that, with respect to the frame on M' defined by $X_1, \cdots, X_{n+1}$, the cocycle for the Γ' action does not depend on x, so it defines a homomorphism A: $\Gamma' \to GL_{n+1}\mathbb{R}$, which must be of the form

$$A = \begin{pmatrix} A_1 & * \\ 0 & \pm 1 \end{pmatrix} \text{ or } \begin{pmatrix} \pm 1 & * \\ 0 & A_1 \end{pmatrix}.$$

Let $\pi : \Gamma' \to \Gamma$ denote the projection. We wish to show that after postcomposition with an automorphism of $GL_{n+1}\mathbb{R}$, there exist homomorphisms $\epsilon_1, \epsilon_2\colon \Gamma' \to \{\pm 1\}$ such that, for all $\gamma' \in \Gamma'$, we have

$$A(\gamma') = \begin{pmatrix} \epsilon_1(\gamma')\pi(\gamma') & 0 \\ 0 & \epsilon_2(\gamma') \end{pmatrix}.$$

(We thank S. Adams for providing the details of the remaining argument.)

Let H' denote the Zariski closure of the image of the map A: $\Gamma' \to GL_{n+1}$ and let $H := H'/A(\ker \pi)$. The composition of A: $\Gamma' \to H'$ with the projection $H' \to H$ gives a map that factors to a map $\Gamma \to H$. According to Proposition

3.1, part 2, $A(\Gamma')$ is not finite. It follows that $H_{\mathbb{R}}$ is not finite. By [M, Theorem 6.15(i)(a), p. 332], H is semisimple. Let Γ^0 denote the preimage of the identity component H^0 of H; Γ^0 is normal in Γ.

Let $L := H^0/Z(H^0)$ denote the quotient of H^0 by its center. It is a consequence of [Z3, Theorem 3.8] that no $\mathbb{R}$-simple factor of L is $\mathbb{R}$-anisotropic. By [M, Theorem 5.6, conclusion (b), p. 228] (applied to each $\mathbb{R}$-simple factor of L), the map $\Gamma^0 \to L_{\mathbb{R}}$ extends to a continuous homomorphism $\mathrm{SL}_n\mathbb{R} \to L_{\mathbb{R}}$. This, in turn, extends to an $\mathbb{R}$-homomorphism $\mathrm{SL}_n \to L$. Since SL_n is algebraically simply connected, we may lift this to a homomorphism $\mathrm{SL}_n \to H^0$. Composing this with the inclusion $H^0 \subseteq H'/A(\ker\pi)$ and applying algebraic simple connectedness again, we lift to an $\mathbb{R}$-homomorphism $\mathrm{SL}_n \to H' \subseteq \mathrm{GL}_{n+1}$.

By the classification of representations of infinite image of SL_n in $n+1$ dimensions, we see that, after post composing with an automorphism of GL_{n+1}, the image of SL_n in GL_{n+1} lies in the diagonal blocked matrix group $\mathrm{SL}_n \times \{e\}$. And, consequently, we may assume that $A(\Gamma')$ lies in the normalizer in $\mathrm{GL}_{n+1}\mathbb{R}$ of $\mathrm{SL}_n\mathbb{R} \times \{e\}$. This normalizer is the group $N := \mathrm{GL}_n\mathbb{R} \times \mathrm{GL}_1\mathbb{R}$. The composition of $A : \Gamma' \to N$ followed by the projection of the reductive group N to its center has, by Kazhdan's property, precompact image. It follows that $A(\Gamma')$ lies in the diagonal blocked matrix group $[\{\pm 1\}\mathrm{SL}_n\mathbb{R}] \times \{\pm 1\}$.

Let B denote the composition of A with the projection $[\{\pm 1\}\mathrm{SL}_n\mathbb{R}] \times \{\pm 1\} \to \mathrm{SL}_n\mathbb{R}$. Note that $B(\Gamma')$ is not finite. We wish to show that, after postcomposing B with an automorphism of GL_n, there exists a homomorphism $\epsilon : \Gamma' \to \{\pm 1\}$ such that, for all $\gamma' \in \Gamma'$, we have $B(\gamma') = \pi(\gamma')\epsilon(\gamma')$.

We now play the same game over again. The map $\Gamma' \to \mathrm{SL}_n/B(\ker\pi)$ factors to Γ. The Zariski closure H_1 of the image is infinite and the adjoint group of the connected component has no $\mathbb{R}$-anisotropic factors; we may therefore apply [M, Theorem 5.6, conclusion (b), p. 228] once again. This time, however, comparison of dimensions shows that the image of Γ in $\mathrm{SL}_n/B(\ker\pi)$ is Zariski dense. It follows that B has Zariski dense image. As $\ker\pi$ is normal in Γ' we conclude that $B(\ker\pi)$ is normal in SL_n.

We thus see that the map $B : \Gamma' \to \mathrm{SL}_n$ factors to a map C: $\Gamma \to \mathrm{PSL}_n$ with Zariski dense image. And, as we have seen, C extends to a (surjective) $\mathbb{R}$-homomorphism $\mathrm{SL}_n \to \mathrm{PSL}_n$. By postcomposing B with an automorphism of SL_n, we may assume that this extension of C is the natural map $\mathrm{SL}_n \to \mathrm{PSL}_n$.

We now have two maps $B : \Gamma' \to \mathrm{SL}_n\mathbb{R}$ and $\pi : \Gamma' \to \pi \subseteq \mathrm{SL}_n\mathbb{R}$. The maps obtained by composing B and π with the projection $SL_n\mathbb{R} \to PSL_n\mathbb{R}$ coincide. Therefore, by multiplying B by a homomorphism $\epsilon : \Gamma' \to \{\pm 1\}$, we may assume that $B = \pi$, and we are done. ∎

4. Proof of Proposition 0.3

For the rest of this paper, we shall assume all the conditions (and consequences) of Lemma 3.4 items (1) and (2). We shall focus attention on the action of a single diffeomorphism $\gamma \in \mathcal{A}$, which we choose to have the following properties: The Oseledec decomposition for γ (which is a C^r decomposition) is $TM = E_1 \oplus \cdots \oplus E_{n+1}, E_i = \mathbb{R}X_i$, and $T\gamma X_i = \pm e^{\chi_i} X_i \circ \gamma$, where

$$\chi_{n+1} = 0,\ \chi_n < 0,\ 0 < \chi_1 < \cdots < \chi_{n-1},\ \chi_1 + \cdots + \chi_n = 0.$$

That such γ can be found follows from Lemmas 3.2 and 3.4. Since the vector fields X_i (defined on M up to sign) commute pairwise, we have integrable subbundles: $E^0 \overset{\text{def}}{=} E_{n+1}$, $E^+ \overset{\text{def}}{=} E_1 \oplus \cdots \oplus E_{n-1}$, $E^- \overset{\text{def}}{=} E_n$, $E^{0+} \overset{\text{def}}{=} E^0 \oplus E^+$, $E^{0-} \overset{\text{def}}{=} E^0 \oplus E^-$, and $E \overset{\text{def}}{=} E_1 \oplus \cdots \oplus E_n$. The corresponding foliations will be denoted: $\mathcal{E}^0$, $\mathcal{E}^+$, $\mathcal{E}^-$, $\mathcal{E}^{0+}$, $\mathcal{E}^{0-}$, respectively. So $(\mathcal{E}^0, \mathcal{E})$, $(\mathcal{E}^+, \mathcal{E}^{0-})$, $(\mathcal{E}^-, \mathcal{E}^{0+})$ are pairs of transversal, C^r foliations. Let $\mathcal{F}$ denote any of the above foliations and $\mathcal{F}(x)$ the leaf of $\mathcal{F}$ through $x \in M$. If B is a subset of M and $x \in B$, denote by $\mathcal{F}_B(x)$ the path connected component of $\mathcal{F}(x) \cap B$ that contains x.

By a **foliation box** we mean an open set $B \subset M$ for which there exists a homeomorphism $h : D^+ \times D^- \times D^0 \to B$, where $D^0 = D^- =$ the open unit interval in $\mathbb{R}$, $D^+ =$ the open unit disc in $\mathbb{R}^{n-1}$, and the leaves of the foliations by discs D^+, D^-, D^0 in $D^+ \times D^- \times D^0$ correspond under h to the leaves of $\mathcal{E}^+$, $\mathcal{E}^-$, and $\mathcal{E}^0$. For all $x \in B$, $y \in \mathcal{E}_B(x)$, and $z \in \mathcal{E}^0_B(x)$, define

$$\mathcal{H}^0_{x,y}(z) \overset{\text{def}}{=} \text{the unique point in } \mathcal{E}_B(z) \cap \mathcal{E}^0_B(y).$$

$\mathcal{H}^0_{x,y} : \mathcal{E}^0_B(x) \to \mathcal{E}^0_B(y)$ is a C^r-diffeomorphism. Similarly define for $x \in B$, $y \in \mathcal{E}^{\pm 0}(x)$, $\mathcal{H}^\pm_{x,y} : \mathcal{E}^\pm_B(x) \to \mathcal{E}^\pm_B(y)$ and for $y \in \mathcal{E}^0(x)$, $\mathcal{H}_{x,y} : \mathcal{E}_B(x) \to \mathcal{E}_B(y)$. For a fixed $x \in B$, these maps define C^r projections π^ν_x: $B \to \mathcal{E}^\nu_B(x)$, $\nu = +, 0, -$, and

$$(\pi^+_x, \pi^-_x, \pi^0_x) : B \to \mathcal{E}^+_B(x) \times \mathcal{E}^-_B(x) \times \mathcal{E}^0_B(x)$$

is a C^r diffeomorphism. If M' is a covering of M, the lift to M' of the above foliations will continue to be denoted by the same symbols.

LEMMA 4.1: *Let M' be any finite covering of M. Then M' contains a dense set of points $\mathcal{P}$ with the following property: For any $x \in \mathcal{P}$ there exists $k = k(x) \in \mathbb{N}$ such that $\gamma^k(\mathcal{E}^0(x)) = \mathcal{E}^0(x)$.*

Proof: This fact follows from a standard dynamical argument, which we reproduce here. First note that for any $X \in E^+(x)$ and $Y \in E^-(x)$,

$$\|T\gamma_x^{-1}X\| \leq \lambda\|X\|, \ \lambda = e^{-\min\{\chi_1,\ldots,\chi_{n-1}\}} < 1$$

$$\|T\gamma_x Y\| \leq \mu\|Y\|, \mu = e^{\chi_n} < 1$$

Here $\|\cdot\|$ is the norm associated to the Riemannian metric for which $\{X_i\}$ is orthonormal. In particular, if $y \in \mathcal{E}^+(x)$ and $d(x,y)$ denotes the distance function determined by integrating $\|\cdot\|$ we have

$$d(\gamma^{-k}x, \gamma^{-k}y) \leq \lambda^k d(x,y), \quad \text{if } y \in \mathcal{E}^+(x),$$

$$d(\gamma^k x, \gamma^k y) \leq \mu^k d(x,y), \quad \text{if } y \in \mathcal{E}^-(x).$$

Poincaré's recurrence theorem ([W]) asserts that for any measurable $S \subset M$ and almost all $x \in S$, there are infinitely many $n \in \mathbb{N}$ such that $\gamma^{-n}(x) \in S$. Let B be a foliation box, $U_{2\epsilon}(x_0)$ a closed ball of radius $2\epsilon > 0$ centered at x_0, entirely contained in B. Let $x \in U_\epsilon(x_0)$ for which we can find $n_1, n_2, \cdots \in \mathbb{N}$ so that $\gamma^{-n_i}(x) \in U_\epsilon(x_0)$. Since γ^{-1} contracts distances in $\mathcal{E}^+$, we can find n_i sufficiently large so that $\gamma^{-n_i}(U_{2\epsilon}(x_0) \cap \mathcal{E}_B^+(x_0))$ has diameter $\leq \epsilon$, hence it is contained in $U_{2\epsilon}(x_0)$. By projecting points of $\gamma^{-n_i}(U_{2\epsilon}(x_0) \cap \mathcal{E}_B^+(x_0))$ into $U_{2\epsilon}(x_0) \cap \mathcal{E}_B^+(x_0)$ via $\pi_{x_0}^+$ we have constructed a continuous map of a closed disc in $\mathcal{E}_B^+(x_0)$ into itself. But such a map must have a fixed point, say $y \in B$, for which $\gamma^{-n_i}(\mathcal{E}^{-0}(y)) = \mathcal{E}^{-0}(y)$, or

$$\gamma^{n_i}(\mathcal{E}^{-0}(y)) = \mathcal{E}^{-0}(y).$$

A similar argument shows that there exists $z \in B$ and $k \in \mathbb{N}$ for which

$$\gamma^k(\mathcal{E}^{+0}(z)) = \mathcal{E}^{+0}(z).$$

If q denotes the minimum common multiple of k and n_i, and $x \in B$ is any point in $\mathcal{E}^{+0}(z) \cap \mathcal{E}^{-0}(y)$, then $\gamma^q(\mathcal{E}^0(x)) = \mathcal{E}^0(x)$. Since the size of B can be chosen arbitrarily small, the lemma follows. ∎

LEMMA 4.2: *M is a fiber bundle over a flat n-dimensional torus with Γ-invariant fibers which are diffeomorphic to $\mathbb{T}^1$, so that the action factors through the quotient $M/\mathbb{T}^1 = \mathbb{T}^n$.*

Proof: Let M' be the covering of M given in Lemma 3.4 item (2). We first show that M' is a $\mathbb{T}^1$-principal bundle over a closed manifold N whose fibers are Γ-invariant. We note that the vector field X^0, denoted X_{n+1} in that lemma, is Γ-invariant, so the main point here is to verify that its integral lines (the leaves of $\mathcal{E}^0$) close up into circles.

Denote by $\varphi_t^Y\colon M' \to M'$ the flow of $Y = a_1X_1 + \cdots + a_nX_n + a_0X^0$, where a_0, a_i are constants. Also define the volume form $\omega = X_1^* \wedge \cdots \wedge X_n^* \wedge X^{0*}$, and the form $\nu = X_1^* \wedge \cdots \wedge X_n^*$, ν being a volume form on the leaves of $\mathcal{E}$. The connection ∇, the foliations, ω, and ν are all φ_t^Y-invariant, a fact that readily follows from the commutation of the above vector fields. Let D denote an open disc contained in a leaf of $\mathcal{E}$. Poincaré's recurrence theorem implies that almost all $x \in D$ (say, with respect to ν) is a recurrent point for the flow $\varphi_t = \varphi_t^{X^0}$. It follows from the existence of a recurrent point that all points of D are recurrent. In fact, if $x \in D$ and if for $t_1, t_2, \cdots \to \infty$ we have $\varphi_{t_i}(x) \in D$ and $\varphi_{t_i}(x) \to x$, then for any vector field Y as above and $y \overset{\text{def}}{=} \varphi_s^Y(x)$,

$$\varphi_{t_i}(y) = \varphi_{t_i}(\varphi_s^Y(x)) = \varphi_s^Y(\varphi_{t_i}(x)) \to \varphi_s^Y(x) = y.$$

In particular, due to Lemma 4.1, there is a recurrent point x of φ_t in D and a number $k = k(x) \in \mathbb{N} \setminus \{0\}$ for which $\gamma^k(\mathcal{E}^0(x)) = \mathcal{E}^0(x)$. For such a point one has $\gamma^k(x) = \varphi_\tau(x)$ for some $\tau \in \mathbb{R}$. Define the diffeomorphism $h = \varphi_{-\tau} \circ \gamma^k$. Then h has the following properties:

(1) $h(x) = x$, $h : \mathcal{E}(x) \to \mathcal{E}(x)$ is a diffeomorphism,

(2) $X_i^h = \pm\lambda_i X_i$, where $\lambda_i = e^{k\chi_i}$ (since φ_t preserves X_i, and γ has the claimed property)

(3) $h \circ \varphi_s = \varphi_s \circ h$ for all $s \in \mathbb{R}$ (since γ preserves X_0). We claim that there exists $t_0 \in \mathbb{R}$ for which $\varphi_{t_0}(x) = x$. In fact let $\varphi_{t_i}(x)$ be a sequence converging to x. Then $\varphi_{t_i}(x) = \varphi_{t_i}(hx) = h\varphi_{t_i}(x)$. So we have a sequence of points converging to x, all of which are fixed by h. On the other hand (2) implies that any fixed point of $h|_{\mathcal{E}(x)}$ is hyperbolic. But a hyperbolic fixed point is isolated (a consequence of Grobman–Hartman's Theorem [P-M]). Therefore for some t_i we have $\varphi_{t_i}(x) = x$. (We learned this nice little trick from A. Katok.)

Let t_0 be the smallest positive number for which $\varphi_{t_0}(x) = x$. Since φ_{t_0} commutes with any other φ_s^Y, it follows that φ_{t_0} is the identity (and t_0 the smallest positive number for which for any other y $\varphi_{t_0}(y) = y$). Denote $\mathbf{T}^1 = \mathbf{R}/t_0\mathbf{Z}$. Then $\mathbf{T}^1$ acts freely on M' according to the expression

$$(t(mod\ t_0), x) \in \mathbf{T}^1 \times M' \to \varphi_t(x) \in M',$$

and the orbits of this action are Γ'-invariant. By projecting under the covering map p: $M' \to M$ we see that M itself is fibered by Γ-invariant circles (although the action of $\mathbf{T}^1$ may no longer be defined in M since X^0 may be defined only up to sign). h factors through the quotient and, due to (2) above, it defines there an Anosov diffeomorphism. It follows from [F] that $M/\mathbf{T}^1$ is homeomorphic to a torus, hence diffeomorphic to a ∇-flat torus.

The existence of a fixed point for the action of Γ on $M/\mathbf{T}$ follows from [H, Theorem 2.22] (in the terminology of that paper, we have shown that this is a **Cartan action**). Therefore Γ fixes a fiber of $M/\mathbf{T}$. According to Theorem 0.1, the action of Γ on the fixed fiber must be finite. So there exists a finite index subgroup of Γ which fixes a point in M (in fact, the whole fiber). ∎

Proposition 0.3 follows now from Lemmas 3.4 and 4.2.

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